On the Scaling Laws of Multi-modal Wireless Sensor Networks

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***Abstract*— In this paper, we consider dense wireless sensor networks deployed to observe *mutiple* random processes. The requirement is to reconstruct an estimate of each random process at the corresponding *collector node*. This leads to multiple many-to-one data gathering wireless channels that interfere with one another. We derive the transport capacity that the network can provide to each process and characterize an achievable rate region for the dense multi-modal network. We further investigate the number of processes that can be observed *simultaneously* by the network. Specifically, we show that it is possible to observe $O\left(N^{\beta}\right)$ processes simultaneously such that the transport capacity scales as $\Theta\left(\log\left(N\right)\right)$ for each of the observed processes, with a large number of sensors N and a fixed total average power. We show this result using a simple scheme based on *antenna sharing*, similar to that proposed in [1]. We then proceed to show that it is possible to simultaneously observe $O\left(N^{\beta}\right)$ continuous, *spatially bandlimited* Gaussian processes using a *fixed* total average power, through a scheme composed of single dimensional quantization, distributed Slepian-Wolf source coding, and the proposed antenna sharing strategy.**

I. Introduction

The capacity of large scale ad-hoc wireless networks has been shown to scale as $\Theta(\sqrt{N})$ as the number of nodes N per unit area grows to infinity [2]. This result advises against the deployment of dense ad-hoc networks since the capacity per node goes to zero as $N \rightarrow \infty$. However, for dense wireless sensor networks, the situation is different because of the increasing correlation in the traffic generated by the nodes as N grows. Some sensing applications require the information observed by the sensors to be transported to a central collector node (*many-to-one* channel), which has the ability to process all the observed information and produce an estimate of the entire random field (process) observed by the sensors. For such applications, the transport capacity was shown to scale as $\Theta\left(1\right)$ in [3], where a constraint is placed on the maximum bit rate. Recently, it was shown in [1] that this result is over-restrictive and the transport capacity was shown to scale as $\Theta\left(\log\left(N\right)\right)$ as $N \rightarrow \infty$, when the bit rate constraint is replaced by a total *power constraint* on the network, which is more appropriate for wireless networks.

In this paper, we investigate a scenario similar to [1], wherein the sensors simultaneously observe a number of processes and wish to transport the observed information to distinct collector nodes, each of which is capable of reproducing one of the observed processes. We will refer to this network as the *multi-modal* sensor network. We derive the transport capacity that the network can provide to each process and characterize an achievable rate region for the network. We further investigate the number of processes that can be observed simultaneously by the network. We first show that it is possible to observe $O\left(N^{\beta}\right)$ processes simultaneously in a dense multi-modal network, with a transport capacity of $\Theta\left(\log(N)\right)$ for each process as $N \rightarrow \infty$. Through a constructive approach, we show that, contrary to the many-to-one scenario considered in [1] and [3], spatial reuse of the bandwidth factors prominently in the multi-modal case. We then proceed to show that it is possible to observe $O\left(N^{\beta}\right)$ spatially bandlimited Gaussian processes simultaneously, as $N \rightarrow \infty$, using a simple *separation* scheme composed of single dimensional quantization, Slepian-Wolf distributed source coding, and the proposed antenna sharing approach.

The rest of the paper is organized as follows. In Section II, we present our modeling assumptions and introduce some notation that will be used throughout the paper. Our results pertaining to the per-process transport capacity of the multi-modal network are developed in Section III. An achievable rate region for the dense multi-modal network is characterized in Section IV. In Section V, we present our main results regarding the observability of continuous spatially bandlimited Gaussian random processes by dense multi-modal wireless networks. Finally, we offer some concluding remarks in Section VI.

II. System Model and Assumptions

We consider the scenario where N sensor nodes are distributed uniformly over the perimeter of a circle with unit radius[1]. Here the network is assumed to be dense (i.e., $N \rightarrow \infty$). Each sensor is capable of simultaneously observing multiple random processes. M collector nodes ($M < N$) are placed on the axis along the center of the circle. Each collector node is interested in gathering information about a different process and the collector nodes do not cooperate with each other. Thus M processes can be simultaneously observed by the sensors. We denote the set of sensor nodes as $\{1, ..., N\}$ and the set of collector nodes as $\{N+1, ..., N+M\}$. We will refer to the distance between the i^{th} and j^{th} nodes as $d_{i,j}$, where $i, j \in \{1, ..., N+M\}$. The distance of the $(N+k)^{th}$ collector node from each of the sensor nodes is denoted as

[1]We realize that this pertains to a specific case. However, we use this restriction to simplify the analysis.

d_k, i.e., $d_{i,N+k} = d_k, \forall i \in \{1,\ldots,N\}$. We further denote the maximum distance between the collector nodes and the set of sensor nodes as d_{max}, i.e., $d_k \leq d_{max}, \forall k \in \{1,\ldots,M\}$. We assume that the sensor nodes are equipped with receivers. We use a discrete time model where $y_j[l]$, the complex-valued signal received by the j^{th} node at time l ($j \in \{1,...,N+M\}$), is given by

$$y_j[l] = \sum_{k=1}^{M} \sum_{\substack{i\in\{1,\ldots,N\} \\ i\neq j}} \frac{e^{j\theta_{i,j}} x_{k,i}[l]}{d_{i,j}^{\delta}} + n_j[l], \qquad (1)$$

where $x_{k,i}[l]$ is the complex-valued signal corresponding to the k^{th} process, transmitted by sensor node i at time l, δ is the path loss exponent assumed to be strictly larger than zero, $\theta_{i,j}$ is the phase shift resulting from the propagation delay, and $n_j[l]$ is the zero mean, unit variance additive Gaussian noise sample received at time l by receiver j. The noise samples are assumed to be spatially and temporally independent. The phase shift between the i^{th} and j^{th} nodes ($\theta_{i,j}$) is assumed to be uniformly distributed in $[0, 2\pi]$ and is independent $\forall i, j$. Moreover, it is assumed to be known at both nodes i and j. In practice, these parameters can be estimated at a marginal loss in throughput (the loss goes to zero as the time scale of the network operation goes to infinity). In (1), it is assumed that all the sensor nodes are synchronized with a common clock. We further assume that the network operates in slotted frames where the duration of one slot T_s is long enough to allow for invoking the asymptotic additive white Gaussian noise (AWGN) channel capacity theorem. Without loss of generality, we focus our analysis on an arbitrary time slot. The path loss model used in (1) implicitly assumes that all the nodes, including the collector nodes, use identical omnidirectional antennas.

We denote the random variable corresponding to the i^{th} process, observed by sensor node j at time l as $u_{ij}[l]$. Thus at time l, sensor node j observes $u_{1j}[l], u_{2j}[l], \ldots, u_{Mj}[l]$. For the class of continuous random processes, we only consider temporally stationary and bandlimited processes. We assume that each process is sampled at the Nyquist rate such that $u_{ij}[l_1]$ and $u_{ij}[l_2]$ are independent and identically distributed for any $l_1 \neq l_2$ and arbitrary i and j. The random variables corresponding to different random processes are also assumed to be independent at each sensor node, i.e., $u_{mj}[l_1]$ and $u_{nj}[l_2]$ are independent for any $m \neq n$ and arbitrary j, l_1 and l_2. The spatial observations $u_{im}[l]$ and $u_{in}[l]$ of process i are, however, correlated for any arbitrary i and l. In fact, the spatial correlation between the observations at adjacent nodes is expected to grow as the density of sensors increases. We assume that the joint distribution of all the observations is known *a-priori* at all sensor nodes. This assumption facilitates the use of distributed source coding in the Slepian-Wolf sense as discussed in the sequel.

We place a constraint on the total average power consumed by the network, i.e.,

$$\frac{1}{T_s} \sum_{l=1}^{T_s} \sum_{i=1}^{M} \sum_{j=1}^{N} |x_{i,j}[l]|^2 \leq P_{total}, \qquad (2)$$

where P_{total} is the total average power assigned to the network[2], which is finite. Together with the finite bandwidth of the shared wireless medium, we believe that this is a faithful representation of the constraints imposed by the wireless channel.

III. The Per-process Transport Capacity

In this section, we assume that the information streams generated by each process are spatially independent at the different sensor nodes. The implications of the spatial correlation between the observations are investigated in Section V. In this context, we define the transport capacity of the multi-modal sensor network with respect to (w.r.t) a particular process observed by the network.

Definition 1: The transport capacity of the network w.r.t to the k^{th} process $C_N(k)$, is defined as the maximum number of bits, belonging to the k^{th} process, that can be transported from the N source nodes to the k^{th} collector node per unit time[3].

Definition 2: We say that $C_N(k) = \Theta(\log(N))$ if there are strictly positive constants c_1, c_2 such that, as $N \to \infty$,

$$c_1 \log(N) \leq C_N(k) \leq c_2 \log(N). \qquad (3)$$

Now we prove the following result that characterizes the scaling law of the per-process transport capacity of dense multi-modal sensor networks, outlined in Section II.

Theorem 3: The transport capacity of the multi-modal network, outlined in Section II, w.r.t each of the observed processes is $\Theta(\log(N))$.

Proof: Without loss of generality, we consider only the transmission of the k^{th} process. Now, the multi-modal network simplifies to a many-to-one network transmitting only the k^{th} process. The transport capacity of the many-to-one sensor network is known [1] to be $\Theta(\log(N))$. Thus the transport capacity provided by the multi-modal network to each process k is

$$C_N(k) = \Theta(\log(N)). \qquad \blacksquare$$

IV. An Achievable Rate Region

We now characterize an achievable rate region for the dense multi-modal sensor network.

Theorem 4: An achievable rate region for the dense multi-modal sensor network ($N \to \infty$) is given by $R_k = \frac{\alpha_k}{2}\log(N)$ with

$$\alpha_k \leq \frac{2\delta}{2\delta+1} \log_N\left(\frac{N}{A}\right) + \log_N\left(\frac{P_k}{P_{total}}\right), k \in \{1,\ldots,A\}$$

[2] P_{total} is normalized to refer to the total average received power at a unit distance from the transmitter.

[3] In our terminology, unit time refers to the duration of transmission of a single symbol.

where R_k denotes the throughput provided to the k^{th} process, P_k denotes the power allocated for the transmission of the k^{th} process and A represents the number of active[4] processes.

Proof: To show achievability, we use a simple transmission scheme, similar to that proposed in [1], that exploits the high density of the nodes to facilitate cooperative transmission. For simplicity of presentation, we consider the symmetric scenario where, for each observed process, all the source nodes generate the same amount of information. The result can be extended to arbitrary asymmetric scenarios using the same argument made in [1].

The main idea of the proposed scheme is to allow closely located nodes to cooperate with each other in transmitting information to the collector nodes, which comes at a very small cost for densely deployed networks. Each node distributes its observed information about a particular process to its neighbors. Since all the sensor nodes do not need to compete for the same collector node, the concept of spatial frequency reuse comes into play. Thereby other nodes, which are far from this node, will be allowed to simultaneously transmit their observed information about other processes to their neighbors. In the next time slot, all the nodes which correctly decode the transmissions intended to them, will cooperate to send the information to the corresponding collector node through beamforming.

We assume that at a particular time instant, only A out of the M processes (collector nodes) are active. The k^{th} process ($k \in \{1, 2, \ldots, A\}$) is allocated a power P_k such that $\sum_{k=1}^{A} P_k \leq P_{total}$. The sensor nodes are divided into A sets with the k^{th} set transmitting the k^{th} process. All the sensor nodes transmit using independent circularly symmetric Gaussian codebooks. In the first time slot, a particular node within each set will be assigned the entire power allocated for observing the corresponding process. Without loss of generality, we consider the transmission of process "k" by the nodes in set "k". The entire power P_k is assigned to node "k_1" and the other nodes in the set are only listening. Let $\alpha_k \log(N)$ denote the rate of transmission of the k^{th} process ($\alpha_k > 0$). Then, a node k_j in the set will be able to decode the transmission of node k_1 correctly iff

$$C_{k_1,k_j} = \log\left(1 + \frac{\frac{P_k}{d_{k_1,k_j}^{2\delta}}}{1 + P_{int}}\right) \geq \alpha_k \log(N) \qquad (4)$$

where P_{int} is the interference caused by the nodes in other sets which are simultaneously transmitting to their neighbors.

Let $d_{k_1,k_j} = N^{-\gamma_{k_1,k_j}}$ where γ_{k_1,k_j} is positive. Let d_{min} be the distance of the closest interfering node from node k_j. It is easy to see that $d_{min} \geq \frac{\pi}{2A}$. Assuming that all the interfering nodes are at a distance d_{min} from node k_j, the interference experienced by the node k_j can be bounded as

$$P_{int} \leq \frac{1}{d_{min}^{2\delta}} \sum_{\substack{m=1 \\ m \neq k}}^{A} P_m \leq A^{2\delta} P_{total}.$$

[4] A process j is said to be *active* if the power allocated to it $P_j \neq 0$.

Thus, we get a sufficient condition for the node k_j to successfully decode the transmission of node k_1 to be

$$C_{k_1,k_j} \geq \log\left(1 + \frac{P_k N^{2\delta\gamma_{k_1,k_j}}}{1 + A^{2\delta} P_{total}}\right) \geq \alpha_k \log(N)$$

$$\Rightarrow \quad \log\left(\frac{P_k N^{2\delta\gamma_{k_1,k_j}}}{A^{2\delta} P_{total}}\right) \geq \alpha_k \log(N)$$

$$\Rightarrow \quad 2\delta\gamma_{k_1,k_j} \geq \alpha_k - \log_N\left(\frac{P_k}{A^{2\delta} P_{total}}\right)$$

$$\Rightarrow \quad \gamma_{k_1,k_j} \geq \frac{\alpha_k}{2\delta} - \frac{1}{2\delta}\log_N\left(\frac{P_k}{A^{2\delta} P_{total}}\right). \qquad (5)$$

Hence all nodes within a distance of $N^{-\gamma_{k_1,k_j}}$ will be able to decode the transmission of node k_1 successfully. Thus the number of nodes that successfully decode the transmission of node k_1 can be lower bounded by $\frac{1}{2\pi} N^{(1-\gamma_{k_1,k_j})}$.

In the second time slot, all the nodes in the set k, which had successfully decoded the transmission of node k_1 in the first time slot, cooperate with node k_1 in a beamforming configuration, with equal power assigned to every node, to deliver the information to collector node "$N + k$" (for all $k \in \{1, \ldots, A\}$). A sufficient condition for the $(N + k)^{th}$ collector node to successfully decode the transmissions of the nodes in the k^{th} set is (see Appendix)

$$\log\left(1 + \frac{\frac{P_k N^{(1-\gamma_{k_1,k_j})}}{2\pi d_{max}^{2\delta}}}{1 + P_{int}}\right) \geq \alpha_k \log(N),$$

where

$$P_{int} = \sum_{\substack{m=1 \\ m \neq k}}^{A} P_m \quad \leq P_{total}.$$

$$\Rightarrow \quad \log\left(\frac{P_k N^{(1-\gamma_{k_1,k_j})}}{2\pi d_{max}^{2\delta} P_{total}}\right) \geq \alpha_k \log(N)$$

$$\Rightarrow \quad 1 - \gamma_{k_1,k_j} + \log_N\left(\frac{P_k}{P_{total}}\right) \geq \alpha_k.$$

In the best case (when γ_{k_1,k_j} satisfies (5) with equality), we get

$$1 - \frac{\alpha_k}{2\delta} + \frac{1}{2\delta}\log_N\left(\frac{P_k}{A^{2\delta} P_{total}}\right) + \log_N\left(\frac{P_k}{P_{total}}\right) \geq \alpha_k$$

$$\Rightarrow \quad \alpha_k \leq \frac{2\delta}{2\delta + 1}\log_N\left(\frac{N}{A}\right) + \log_N\left(\frac{P_k}{P_{total}}\right). \qquad (6)$$

Since it took two time slots to deliver $\alpha_k T_s \log(N)$ bits of the k^{th} process to the corresponding collector node, the throughput provided by the proposed scheme to the k^{th} process is $R_k = \frac{\alpha_k}{2}\log(N)$.

The final step is to symmetrize the transmission scheme by assigning every two consecutive time slots to different sets of sensor nodes. For the case when $A = 1$ (the many-to-one channel), the achievability condition (6) reduces to $\alpha_1 \leq \frac{2\delta}{2\delta+1}$, which matches the result obtained in [1]. ■

It is easy to see from (6) that α_k can be positive only if the power allocated for observing the k^{th} process P_k satisfies

$$P_k > P_{total}\left(\frac{A}{N}\right)^{\frac{2\delta}{2\delta+1}} \tag{7}$$

It is important to note that even an equality in (7) makes the transport capacity w.r.t the k^{th} process to scale as $\Theta(1)$ instead of $\Theta(\log(N))$.

We next find the power allocation strategy which maximizes the total achievable rate $\left(\sum_{k=1}^{A}\alpha_k\right)$ using the Lagrange constrained optimization method.

$$J = \sum_k \alpha_k + \lambda(\sum_k P_k - P_{total}).$$

Differentiating w.r.t P_k and equating to 0, we get

$$P_k = -\frac{\log_N(e)}{\lambda}.$$

Using the power constraint, we get

$$\lambda = -\frac{A\log_N(e)}{P_{total}} \qquad \Rightarrow P_k = \frac{P_{total}}{A}.$$

Thus, an equal distribution of power among the observed processes maximizes the total achievable rate. In this case, the achievability condition in (6) reduces to

$$\alpha_k \le \frac{2\delta}{2\delta+1} - \left(\frac{4\delta+1}{2\delta+1}\right)\log_N(A) \quad \forall k = \{1, 2, \ldots, A\} \tag{8}$$

We are now ready to derive the following result regarding the number of processes that can be simultaneously observed by the network.

Theorem 5: The dense multi-modal network can observe N^β identical processes simultaneously, such that the transport capacity w.r.t each process is $\frac{\alpha}{2}\log(N)$, where

$$\beta \le \frac{2\delta}{4\delta+1} - \frac{\alpha(2\delta+1)}{4\delta+1}.$$

Proof: Let $A = N^\beta$. Then the achievability condition in (8) reduces to

$$\alpha \le \frac{2\delta}{2\delta+1} - \left(\frac{4\delta+1}{2\delta+1}\right)\beta.$$

Thus, we obtain an upper bound on the exponent β of the number of processes simultaneously observable by the network as

$$\beta \le \frac{2\delta}{4\delta+1} - \frac{\alpha(2\delta+1)}{4\delta+1}.$$

When $\alpha = 0$, we find that $\beta \le \frac{2\delta}{4\delta+1}$. Thus, it is possible to observe upto $N^{\left(\frac{2\delta}{4\delta+1}\right)}$ processes and achieve $\Theta(\log(N))$ capacity for each process, using the proposed scheme. ■

A few remarks on the achievability results are now in order.

1) If the positions of the sensor nodes are chosen according to a uniform i.i.d assumption, rather than uniformly, it is easy to see that our results hold with high probability as $N \to \infty$.
2) Our path loss model is based on the far field wave propagation assumption. We realize that this model may not be very accurate since the far field assumption does not hold when the transmitter and receiver are very close to each other. Thus, using more refined path loss models is a possible venue for future work. However, even with more refined models, one would still expect that, in dense networks, every node can distribute its information to its neighbors at minimal cost. Hence, we conjecture that the main idea of utilizing the proximity of source nodes in dense networks to facilitate efficient cooperation protocols will play a significant role with these models as well.
3) It was shown in [1] that spatial resuse does not factor prominently in the *many-to-one* scenario since it does not resolve the competition for the same destination, which is the dominant factor that dictates the $\Theta(\log(N))$ scaling law of the total transport capacity. However, in the multi-modal case, the availability of multiple collector nodes resolves this issue and hence spatial reuse factors prominently, as is evident from the proof of Theorem 3, wherein different sets of nodes are allowed to reuse the bandwidth for transmitting to different collector nodes. Infact the spatial frequency reuse results in an increased **total** transport capacity of $O\left(N^\beta \log(N)\right)$ for the proposed scheme, as opposed to a total transport capacity of $\Theta(\log(N))$ for the many-to-one scenario.
4) The scheme we have proposed is simple in the sense that the receivers treat all the interference as noise. However, we do not claim that our scheme achieves the optimal total transport capacity of the multi-modal network. It may be possible to observe more number of processes using sophisticated multi-user detection schemes, where the receivers have more interference cancellation capabilities.

V. The Observability of Spatial Random Processes

In this section, we investigate the effect of the correlation between the observations of a particular process at the different sensors on the operation of wireless multi-modal sensor networks. Specifically, we use Theorems 3 and 5 along with the necessary and sufficient conditions for the observability of random processes, established in [1], to show that it is possible to observe $O\left(N^\beta\right)$ Gaussian spatially bandlimited processes with only finite total average power. Throughout this section, we focus on the asymptotic scenario as $N \to \infty$. For completeness, we repeat the definitions and the conditions of observability from [1].

Definition 6: A discrete random sequence is said to be observable if it can be **detected** at the collector node with arbitrarily small probability of error for a certain allocation of a finite bandwidth and a finite total average power to the network.

It was shown in [1] that a spatial random process k is observable iff

$$H\left(u_{k1}[l], u_{k2}[l], ..., u_{kN}[l]\right) \leq \Theta\left(\log(N)\right), \qquad (9)$$

where the random variables $\{u_{kj}[l]\}$ are assumed to be discrete and $H(.,.,.)$ refers to the joint entropy.

Definition 7: A continuous random process is said to be observable if it can be **estimated** at the collector node with a non-zero mean square error for a certain allocation of a finite bandwidth and a finite total average power to the network.

It was shown in [1] that all Gaussian *spatially bandlimited* processes can be observed by dense wireless sensor networks as $N \to \infty$, since the quantized random variables $\{v_{kj}[l]\}$ satisfy the upper bound given in (9). This result was shown to hold even when a source/channel separation scheme, comprising of a lossy source encoder (single dimensional quantization followed by Slepian-Wolf distributed coding) concatenated with the cooperative transmission strategy of [1], is used.

Combining the results of Theorems 3 and 5 with these observability conditions, we establish our *main* result.

Theorem 8: The dense multi-modal network is capable of observing N^{β} identical Gaussian spatially bandlimited processes as $N \to \infty$, where

$$\beta \leq \frac{2\delta}{4\delta+1} - \frac{\alpha(2\delta+1)}{4\delta+1}$$

and α depends on the spatial bandwidth of the process. Moreover, this result holds even if we restrict ourselves to the source/channel separation scheme comprising of a lossy source encoder (single dimensional quantization followed by Slepian-Wolf distributed coding) concatenated with the proposed cooperative transmission strategy.

Proof: From Theorem 5, we know that N^{β} identical processes can be observed simultaneously by the dense multi-modal network, such that the transport capacity w.r.t each of the processes is $\frac{\alpha}{2}\log(N)$, where $\beta \leq \frac{2\delta}{4\delta+1} - \frac{\alpha(2\delta+1)}{4\delta+1}$. It was shown in [3] that for a Gaussian spatially bandlimited process, the joint entropy of the quantized random variables $\{v_{kj}[l]\}$ is given by

$$H\left(v_{k1}[l], ..., v_{kN}[l]\right) \approx \sqrt{2b}\log\left(\frac{cN}{\sqrt{2b}}\right) \quad as \quad N \to \infty,$$

where $b = 4a^2\pi^2\int_{-f_0}^{f_0} f^2 S_R(f)df < \infty$ and c is a constant that depends on the quantization step. Here f_0 is the spatial bandwidth of each of the observed Gaussian processes. Thus, the joint entropy of the quantized random variables satisfies

$$H\left(v_{k1}[l], ..., v_{kN}[l]\right) \leq \frac{\alpha}{2}\log(N),$$

where α is a function of f_0. Since each of the N^{β} processes satisfies the observability condition (9), all of them are observable by the network. ∎

A few remarks are now in order

1) While Theorem 8 is stated for bandlimited processes, clearly the result extends to other processes that generate $\leq \Theta\left(\log(N)\right)$ bits per transmission symbol. The Gaussian process with $R(d) = e^{-d^2}$ given in [3], is one example of such processes.
2) While we have assumed the observed processes to be identical, our results clearly extend to the case of non-identical observed processes which require different transmission rates.

VI. Concluding Remarks

In this paper, we have established the gains possible with cooperation between the sensor nodes. We characterized an achievable rate region for dense multi-modal wireless sensor networks using a scheme that exploits the proximity of sensors to allow for efficient cooperation. We have shown that it is possible to observe $O\left(N^{\beta}\right)$ processes simultaneously, and still achieve a transport capacity of $\Theta\left(\log(N)\right)$ for each of the processes, with a large number of sensors N and a fixed total average power. We then used the necessary and sufficient conditions for observability, from [1], to show that a dense multi-modal sensor network can observe $O\left(N^{\beta}\right)$ spatially bandlimited Gaussian processes simultaneously. Extending these results to fading channels and using more refined path loss models are possible venues for future work.

Appendix

Here we derive an upper bound on the interference experienced at the $(N+k)^{th}$ collector node during the second time slot. During this slot, the nodes in each set m cooperate among themselves to beamform to the $(N+m)^{th}$ collector node. Let x_m ($m \in \{1, \ldots, A\}$) denote the signal transmitted by each of the $\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}$ sensor nodes in the set m, which had successfully decoded the transmission of node m_1 in the first time slot. Now, the signal received at the $(N+k)^{th}$ collector node is given by

$$y_{N+k}[l] = \sum_{\substack{m=1 \\ m\neq k}}^{A} \sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} x_m[l]e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})} + \frac{N^{(1-\gamma_{k_1,k_j})}}{2\pi}x_k[l],$$

where the second term is due to the beamforming by the sensor nodes of the k^{th} set. Since all the transmitted signals $\{x_m\}$ are chosen from a Gaussian codebook and the noise is Gaussian, the received signal y_{N+k} is also Gaussian distributed. The interference experienced at the $(N+k)^{th}$ collector node is given by

$$P_{int} = E\left|\sum_{\substack{m=1 \\ m\neq k}}^{A} \sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} x_m[l]e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})}\right|^2$$

$$= \sum_{\substack{m=1\\m\neq k}}^{A} E\left|\sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} x_m[l]e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})}\right|^2$$

(since each of the terms in the outer summation has zero mean and the terms are independent of each other)

$$= \sum_{\substack{m=1\\m\neq k}}^{A} E\left|x_m[l]\right|^2\left|\sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})}\right|^2.$$

As $N \to \infty$, the law of large numbers suggests that the term

$$\left|\sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})}\right|^2$$

converges in probability to its expected value. Thus the interference at the $(N+k)^{th}$ collector node P_{int} converges in probability, as $N \to \infty$, to

$$\sum_{\substack{m=1\\m\neq k}}^{A} E\left|x_m[l]\right|^2 E\left|\sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})}\right|^2.$$

Hence for large N, the interference P_{int} is given by

$$\sum_{\substack{m=1\\m\neq k}}^{A} E\left|x_m[l]\right|^2 \sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} E\left|e^{j(\theta_{n_m,N+k}-\theta_{n_m,N+m})}\right|^2$$

(since each of the terms in the summation has zero mean and the terms are independent of each other)

$$\Rightarrow P_{int} = \sum_{\substack{m=1\\m\neq k}}^{A} \frac{P_m}{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} \sum_{n_m=1}^{\frac{1}{2\pi}N^{(1-\gamma_{m_1,m_j})}} 1$$

$$\Rightarrow P_{int} = \sum_{\substack{m=1\\m\neq k}}^{A} P_m \leq P_{total}.$$

The transport capacity of the k^{th} process is given by (e.g. [4])

$$C_N(k) = \log\left(1 + \frac{\frac{P_k N^{(1-\gamma_{k_1,k_j})}}{2\pi d_{max}^{2\delta}}}{1+P_{int}}\right)$$

For large N, the capacity can be given as

$$C(k) = \lim_{N\to\infty} C_N(k) = \log\left(1 + \frac{\frac{P_k N^{(1-\gamma_{k_1,k_j})}}{2\pi d_{max}^{2\delta}}}{1+\lim_{N\to\infty} P_{int}}\right)$$

(due to the continuity of the $\log(1+x)$ function). Thus, for large N, the capacity converges in probability to

$$C(k) \to \log\left(1 + \frac{\frac{P_k N^{(1-\gamma_{k_1,k_j})}}{2\pi d_{max}^{2\delta}}}{1+\sum_{\substack{m=1\\m\neq k}}^{A} P_m}\right),$$

which is independent of the values of $\{\theta_{i,j}\}$ of any particular realization.

REFERENCES

[1] H. El Gamal. On the scaling laws of dense wireless sensor networks. *Submitted to IEEE Transactions on Information Theory*, April 2003.

[2] P. Gupta and P.R.Kumar. The capacity of wireless networks. *IEEE Transactions on Information Theory*, 46(2), March 2000.

[3] D. Marco, E. J. Duarte-Melo, M. Liu, and D. L. Neuhoff. On the many-to-one transport capacity of a dense wireless sensor network and the compressibility of its data. In *Information Processing in Sensor Networks*, 2003.

[4] E. Telatar. Capacity of multi-antenna gaussian channels. *European Transactions on Telecommunications*, 10(6):585–595, November 1999.